

# Consensus dynamics

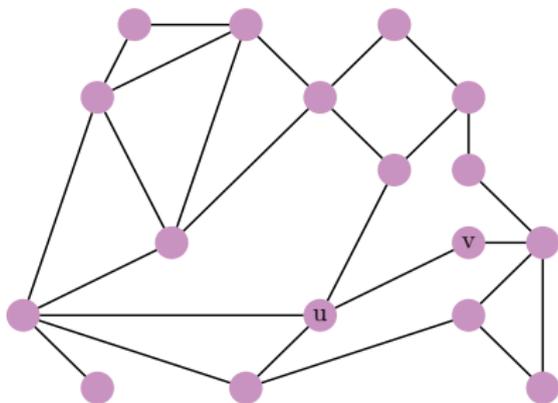
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## Averaging dynamics

$\mathcal{G} = (\mathcal{V}, \mathcal{E})$  (undirected) graph ,



Nodes represent socio-economic agents/sensors,

Edges denote friendship/communication link

$y_v \in \mathbb{R}$  initial opinion (measure) of node  $v$

**Averaging dynamics:**

$$\begin{cases} x_v(t+1) &= \frac{1}{2}x_v(t) + \frac{1}{2d_v} \sum_{u:(u,v) \in \mathcal{E}} x_u(t) \\ x_v(0) &= y_v \end{cases}$$

where  $d_v$  is the degree of node  $v$ .

## Averaging dynamics

$$\begin{cases} x_v(t+1) &= \frac{1}{2}x_v(t) + \frac{1}{2d_v} \sum_{u:(u,v) \in \mathcal{E}} x_u(t) \\ x_v(0) &= y_v \end{cases}$$

In a compact form  $x(t+1) = Px(t)$ .

This yields  $x(t) = P^t y$ .

**Asymptotics:**  $P^t \rightarrow ?$

## Stochastic matrices

$P \in \mathbb{R}^{V \times V}$  above is an example of a **stochastic matrix**:  $P_{uv} \geq 0$ ,  $\sum_v P_{uv} = 1$  for all  $u \in V$ .  $P\mathbb{1} = \mathbb{1}$ .

Given  $P$ , we can consider the underlying graph  $\mathcal{G}_P = (\mathcal{V}, \mathcal{E}_P)$  where  $\mathcal{V}$  is the set of nodes and where the set of edges is given by  $\mathcal{E}_P := \{(u, v) \in \mathcal{V} \times \mathcal{V} \mid P_{uv} > 0\}$

Remarkably, some of the key properties of  $P$  responsible for the transient and asymptotic behavior of  $P^t$  are determined by the connectivity properties of the underlying graph  $\mathcal{G}_P$ .

$\mathcal{G}_P$  is said to be *strongly connected* if for any pair of vertices  $u \neq v$  in  $\mathcal{V}$  there is a path in  $\mathcal{G}_P$  connecting  $u$  to  $v$ .

# Perron-Frobenius theorem

## Theorem

Assume that  $P \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$  is such that  $G_P$  is strongly connected and  $P_{uu} > 0$  for at least one node  $u \in \mathcal{V}$ . Then,

1. 1 is an algebraically simple eigenvalue of  $P$ .
2. There exists a (unique) probability vector  $\pi \in \mathbb{R}^{\mathcal{V}}$  ( $\pi_v > 0$  for all  $v$  and  $\sum_v \pi_v = 1$ ) s.t.  $\pi^* P = \pi^*$ .
3. All the remaining eigenvalues of  $P$  are of modulus  $< 1$ .

Consequence:  $P^t \rightarrow \mathbb{1}\pi^*$  for  $t \rightarrow +\infty$ . This yields

$$\lim_{t \rightarrow +\infty} x(t) = \lim_{t \rightarrow +\infty} P^t x(0) = \mathbb{1}(\pi^* x(0))$$

In other terms dynamics leads asymptotically to a **consensus**: all agents' state converging to the common value  $\pi^* x(0)$ , called consensus point which is a convex combination of the initial states with weights given by the invariant probability components.

## Average consensus

If  $\pi$  is the uniform vector (i.e.  $\pi_u = |\mathcal{V}|^{-1}$  for all  $u$ ), the common asymptotic value is simply the arithmetic mean of the initial states.

In this case all agents equally contribute to the final common state: **average consensus**.

This uniformity condition amounts to assume that  $\mathbb{1}^* P = \mathbb{1}^*$ , namely that also columns of  $P$  sum to 1: a sufficient condition for this being that  $P$  is symmetric.

# Applications

In many applications, uniformity is necessary and is enforced in the model. Indeed, the distributed computation of the arithmetic mean is an important step to solve estimation problems for sensor networks.

## Specific example:

- ▶  $N$  sensors deployed in a certain area;
- ▶ Each of them makes a noisy measurement of a physical quantity  $x$ :  $y_v = x + \omega_v$  where  $\omega_v$  are i.i.d. zero mean Gaussian noise
- ▶ Optimal mean square estimator:  $\hat{x} = N^{-1} \sum_v y_v$ .

More sophisticated estimation problems: quantity to be estimated is time-varying, sensors may have different performances

Other fields of application: opinion dynamics, computer load balancing, control of cooperative autonomous vehicles

## Random walks on graphs

Stochastic matrices owe their name to their use in probability. Indeed, given a stochastic matrix  $P \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$ , the term  $P_{vw}$  can be interpreted as the **probability of making a transition** from state  $v$  to state  $w$ : you can imagine to be sitting at state  $v$  and to walk along one of the available outgoing edges from  $v$  according to the various probabilities  $P_{vw}$ . In this way you construct what is called a random walk on the underlying graph  $\mathcal{G}$ .

The case of uniform transition probabilities  $P_{vw} = \frac{1}{d_v}$  for all  $w$  such that  $(w, v) \in \mathcal{E}$  is said to be the **simple random walk** on  $\mathcal{G}$ .

$\mathcal{G}$  connected undirected  $\Rightarrow \pi_v = \frac{d_v}{\sum_{w \in \mathcal{V}} d_w}$  is the invariant probability (check this!)

## The rate of convergence

Basic linear algebra allows to study the rate of convergence to consensus: it will be dictated by the largest in modulo among the eigenvalues of  $P$  except 1; precisely,

### Theorem

*Let  $P \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$  be a stochastic matrix such that  $\mathcal{G}_P$  is strongly connected and  $P_{uu} > 0$  for some  $u$ . Consider all its eigenvalues  $\mu_i$  but 1 and put  $\rho_2 = \max\{|\mu_i| < 1\}$ . Then, for every  $\epsilon > 0$  there exists a constant  $C_\epsilon$  such that*

$$\|(P^t - \mathbb{1}\pi^*)x_0\|_2 \leq C_\epsilon(\rho_2 + \epsilon)^t \|x_0\|_2 \quad \text{for all } t.$$

The parameter  $\rho_2$ , introduced in the statement of the proposition above, is also called the **second eigenvalue** of  $P$ , and the difference  $1 - \rho_2$  the **spectral gap** of  $P$ .

The above result essentially says that **convergence to consensus** happens **exponentially fast** as  $\rho_2^t$ .

## Average dynamics with stubborn agents

We now investigate consensus dynamics models where some of the agents do not modify their own state (stubborn agents).

These models are of interest in socio-economic models and also in vehicle rendezvous problems where certain vehicles want to remain fixed and make the other gather around them

Consider a symmetric connected graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ .

We assume a splitting  $\mathcal{V} = \mathcal{S} \cup \mathcal{R}$ : agents in  $\mathcal{S}$  are **stubborn** agents not changing their state, agents in  $\mathcal{R}$  are **regular** agents.

Let  $P \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$  be a stochastic matrix such that, for  $u \neq v$

$$P_{uv} = 0 \Leftrightarrow (v, u) \notin \mathcal{E} \text{ or } u \in \mathcal{S} \quad (1)$$

Dynamics of opinions:  $x(t+1) = Px(t)$ .

## Average dynamics with stubborn agents

Order elements in  $\mathcal{V}$  in such a way that elements in  $\mathcal{R}$  come first:

$$P = \begin{bmatrix} Q^{11} & Q^{12} \\ 0 & I \end{bmatrix}$$

Splitting accordingly  $x(t) = (x^{\mathcal{R}}(t), x^{\mathcal{S}}(t)) \in \mathbb{R}^{\mathcal{V}}$

$$\begin{aligned} x^{\mathcal{R}}(t+1) &= Q^{11}x^{\mathcal{R}}(t) + Q^{12}x^{\mathcal{S}}(t) \\ x^{\mathcal{S}}(t+1) &= x^{\mathcal{S}}(t) \end{aligned}$$

$Q^{11}$  is asymptotically stable:  $(Q^{11})^t \rightarrow 0$ .

Henceforth,  $x^{\mathcal{R}}(t) \rightarrow x^{\mathcal{R}}(\infty)$  for  $t \rightarrow +\infty$  and

$$x^{\mathcal{R}}(\infty) = Q^{11}x^{\mathcal{R}}(\infty) + Q^{12}x^{\mathcal{S}}(0)$$

which is equivalent to

$$x^{\mathcal{R}}(\infty) = (I - Q^{11})^{-1}Q^{12}x^{\mathcal{S}}(0)$$

$P$  stochastic implies that asymptotic opinions of regular agents are convex combinations of the opinions of stubborn agents.

## The electrical network interpretation

Assume  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  undirected and think of edges as electrical resistances having resistance equal to 1

- ▶  $\eta \in \mathbb{R}^{\mathcal{V}}$  currents flowing into the network.  $\sum_v \eta_v = 0$ ,
- ▶  $\Phi_{(u,v)}$  current along the edge  $(u, v)$ ,
- ▶  $W_u$  voltage at node  $u$ .
- ▶ **Kirchoff law:**  $\sum_v \Phi_{(u,v)} = \eta_u$
- ▶ **Ohm's law:**  $W_v - W_u = \Phi_{(u,v)}$

Consequence:

$$(I - P)W = D_G^{-1}\eta$$

where  $P$  is the SRW on  $\mathcal{G}$  and  $D_G$  is diagonal with  $(D_G)_{vv} = d_v$ .

## The electrical network interpretation

$$(I - P)W = D_G^{-1}\eta$$

$P$  coincide with  $Q$  in the upper part!

$$(I - P) \begin{pmatrix} x^{\mathcal{R}}(\infty) \\ x^{\mathcal{S}}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ \theta \end{pmatrix}$$

This implies that  $x^{\mathcal{R}}(\infty)$  can be **interpreted as voltages** at the regular nodes when stubborn nodes are kept at fixed voltage!

Techniques of electrical circuits can be used to compute or estimate asymptotic opinions.